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Buoyant turbulent flow driven by internal energy generation

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Abstract—Two thermal microscales for buoyancy driven turbulent flows are proposed. The first of these scales, arranged relative to viscous dissipation, is

$$\eta_\theta \sim \left(1+\frac{1}{\sigma}\right)^{1/4}\left(\frac{\nu a^2}{\mathscr{P}_\beta}\right)^{1/4}$$

which explicitly includes the limit for $\sigma \to \infty$, and, arranged relative to inertial production, is

$$\eta_\theta \sim (1+\sigma)^{1/4}\left(\frac{a^3}{\mathscr{P}_\beta}\right)^{1/4}$$

which explicitly includes the limit for $\sigma \to 0$. Here $\sigma = \nu/a$ denotes the Prandtl number† and $\mathscr{P}_\beta$ the production of buoyant turbulent energy. The limits of this scale for $\sigma \sim 1$, and $\sigma \to 0, \infty$ are shown to be the celebrated Kolmogorov scale and its extensions known as the Oboukhov–Corrsin and Batchelor scales, respectively. The η_θ scale is independent of any integral (or geometric) effect.

The second of these scales in terms of the limit for $\sigma \to \infty$ is

$$\lambda_\theta \sim l^{1/3}\left(1+\frac{1}{\sigma}\right)^{1/6}\left(\frac{\nu a^2}{\mathscr{P}_\beta}\right)^{1/6}$$

and in terms of the limit for $\sigma \to 0$ is

$$\lambda_\theta \sim l^{1/3}(1+\sigma)^{1/6}\left(\frac{a^3}{\mathscr{P}_\beta}\right)^{1/6}$$

where l is an integral (or geometric) scale. When expressed in terms of buoyant force induced by internal energy generation, these scales relative to the integral scale become

$$\eta_\theta/l \sim \Pi_{\mathrm{I}}^{-1/4} \quad \lambda_\theta/l \sim \Pi_{\mathrm{I}}^{-1/6}$$

where

$$\Pi_{\mathrm{I}} \sim \frac{Ra_{\mathrm{I}}}{1+Pr^{-1}}$$

and

$$Ra_{\mathrm{I}} = \frac{g\beta\Phi l^5}{\nu a^2}$$

is the appropriate Rayleigh number and Pr is the Prandtl number. Here $\Phi = u'''/\rho c_p$, u''' being the rate of energy generation per unit volume.

A two-layer heat transfer model for turbulent flow driven by internal energy generated between two horizontal plates is proposed. The model yields, in terms of the foregoing scales,

$$Nu \sim \frac{l/\eta_\theta}{1-(\lambda_\theta/l)^2(l/\eta_\theta)}$$

or, in terms of Π_{I}

†For notational convenience, the Prandtl number is denoted by σ in scale developments. For customary reasons, it is denoted by Pr among the dimensionless numbers for heat transfer.

NOMENCLATURE

a	thermal diffusivity
c_p	specific heat at constant pressure
C_0, C_1, C_2	constants
F_B	buoyancy force
F_I	inertial force
F_V	viscous force
g	gravitational acceleration
g_i	gravitational acceleration vector
Gr	Grashof number
k	thermal conductivity
K	mean kinetic energy
K_θ	rms of temperature fluctuations
l	a characteristic length for geometry
Nu	Nusselt number, $\Phi l^2/a\Delta T$
Pe_N	Peclet number for buoyancy driven flows
Pr	Prandtl number
q	heat flux
q_c	core heat flux
Q_H	enthalpy flow
Q_K	conduction heat flux
Ra	Rayleigh number
Ra_c	critical Rayleigh number
s_{ij}	fluctuating rate of strain
S_{ij}	mean rate of strain
T	temperature
u	root mean square of velocity fluctuation
u_c	core velocity
u_i	velocity fluctuation
u'''	rate of energy generation/volume
U_i	mean velocity
V	a characteristic velocity.

Greek symbols

β	coefficient of thermal expansion
δ_v	momentum boundary layer thickness
δ_t	thermal boundary layer thickness
Δ	difference
ε	viscous dissipation
ε_θ	thermal dissipation
η	Kolmogorov scale
η_θ	thermal microscale
η_θ^B	Batchelor scale
η_θ^C	Oboukhov–Corrsin scale
θ	temperature fluctuation
θ_c	core temperature
Θ	mean temperature
Θ_0	temperature of isothermal ambient
λ	Taylor scale
λ_θ	thermal microscale
μ	dynamic viscosity
ν	kinematic viscosity
$\Pi_{N,I}$	dimensionless number for buoyancy driven flows
ρ	density
σ	Prandtl number
Φ	energy generation (dimensionless).

Script symbols

$\mathscr{D}_i$	mean transport
$(\mathscr{D}_\theta)_j$	thermal transport
$\mathscr{P}$	inertial production
$\mathscr{P}_\beta$	buoyant production
$\mathscr{P}_\theta$	thermal production.

Superscripts

$\tilde{}$	instantaneous value
$\bar{}$	mean value.

$$Nu \sim \frac{\Pi_I^{1/4}}{1-\Pi_I^{-1/12}}$$

where Nu denotes the Nusselt number. The special case of this relation for $Pr > 1$ is already known to correlate the experimental literature on electrolytically heated water.

1. INTRODUCTION

Buoyant turbulent flow driven by internal energy generation has been lately receiving increased attention because of its relevance to post-accident heat removal from nuclear systems. While the turbulent motion associated with the classical Benard problem has been extensively studied (see Arpaci [1] for a latest list of references), the literature on the buoyant flow driven by energy generation is confined only to fewer studies (see, for example, Cheung [2] for a list of references). Furthermore, the microscale foundations of heat and mass transfer in buoyancy driven flows have apparently escaped any attention except for the recent studies by Arpaci [1, 3] and Arpaci and Selamet [4].

The present study as well as the past modeling efforts by Cheung [5] and Cheung *et al.* [6] are all motivated by the experimental work of Kulacki and co-workers [7, 8]. Generalizing the existing models, the present study demonstrates how two Cheung correlations for the limits of $Pr \ll 1$ and $Pr \gg 1$ can be combined into one for any Prandtl number. The study involves five sections and one appendix : following this introduction, Section 2 introduces a fundamental dimensionless number for buoyancy driven flows, Section 3 develops the microscales appropriate for buoyant turbulent flows driven by internal energy generation, Section 4 constructs a heat transfer model in terms of these scales and Section 5 concludes the study. In the appendix, a microscale interpretation of the existing literature is given.

2. A DIMENSIONLESS NUMBER

As is well-known, the independent dimensionless numbers characterizing buoyancy driven flows are the Rayleigh and Prandtl numbers, Ra and Pr, respectively. A dimensionless number recently proposed by Arpaci [3] explicitly describes these flows by a combination of Ra and Pr. A review of this dimensionless number is needed for the microscales of buoyancy driven flows.

Let the buoyancy driven momentum balance be

$$F_B \sim F_I + F_V \tag{1}$$

where F_B, F_I and F_V denoting respectively the buoyant, inertial and viscous forces. Also, let the thermal energy balance be

$$Q_H \sim Q_K \tag{2}$$

where Q_H and Q_K denoting respectively the enthalpy flow and conduction. Then, from equation (1)

$$\frac{F_B}{F_I + F_V} \sim \frac{F_B/F_V}{F_I/F_V + 1} \tag{3}$$

and from equation (2)

$$Q_H/Q_K \tag{4}$$

the numeral 1 in equation (3) implying order of magnitude. Although the force ratios of equation (3) and the energy ratio of equation (4) are dimensionless, they are usually expressed in terms of velocity which is a dependent variable in buoyancy driven flows :

$$\frac{F_B}{F_V} \sim \frac{g(\Delta\rho)l^2}{\mu V} \quad \frac{F_I}{F_V} \sim \frac{\rho V l}{\mu} \quad \frac{Q_H}{Q_K} \sim \frac{\rho c V l}{k} \tag{5}$$

where l is a characteristic length, and the rest of the notation is conventional. Now, the combination of equations (3) and (4) for a result independent of velocity yields

$$\Pi_N \sim \frac{(F_B/F_V)(Q_H/Q_K)}{(F_I/F_V)(Q_K/Q_H) + 1} \tag{6}$$

or

$$\Pi_N \sim \frac{Ra}{1 + Pr^{-1}} = \frac{PrRa}{1 + Pr} \tag{7}$$

which is the appropriate dimensionless number for natural convection in any fluid. Here,

$$\sigma = Pr = \frac{\nu}{a} \quad Ra = \frac{g}{\nu a}\left(\frac{\Delta\rho}{\rho}\right)l^3$$

respectively denote the Prandtl and Rayleigh numbers. The two limits of equation (7) are

$$\lim_{Pr \to 0} \Pi_N \to PrRa = Pe_N$$

Pe_N being a Peclet number for buoyancy driven flows, and

$$\lim_{Pr \to \infty} \Pi_N \to Ra$$

(see, for example, pp. 116–119 of Bejan [9]).

For a specified temperature difference, the definition of the coefficient of isobaric expansion,

$$\beta = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p$$

gives

$$\frac{\Delta\rho}{\rho} \sim \beta\Delta T$$

and Π_N now depends on the usual form of

$$Ra = \frac{g\beta\Delta T l^3}{\nu a}. \tag{8}$$

Although the existence of Π_N has never been directly shown, the integral solution for the laminar natural convection near a vertical plate given by Squire [10] almost half a century ago leads for heat transfer to

$$Nu = 0.508 Pr^{1/2}(Pr + 20/21)^{-1/4} Gr^{1/4}$$

where Gr is the usual Grashof number. Recalling $Ra = GrPr$, this result can be rearranged in terms of Π_N,

$$Nu = 0.508 \Pi_N^{1/4}$$

where Nu is the Nusselt number, and

$$\Pi_N = \frac{Ra}{0.952 + Pr^{-1}}.$$

(Also, see p. 133 of Bejan [9]). Since then the explicit role of Π_N in studies on buoyancy driven flows is usually ignored. For example, an experimental study by Krishnamurti [11] shows the cascade of transitions in buoyancy driven flows past the Benard instability. Any two successive transitions, illustrated here in terms of the first two, can be qualitatively related by a simple model depending on Π_N,

$$(Ra_c)_{II} = (Ra_c)_I + \frac{(\Delta Ra_c)_I^{II}}{1 + Pr^{-1}}$$

or

$$(Ra_c)_{II} = (Ra_c)_I + (\Delta\Pi_N)_I^{II}$$

where

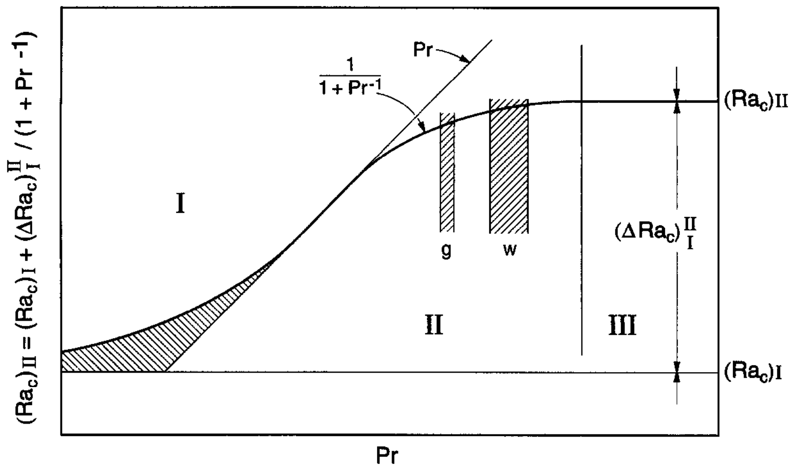

Fig. 1. A sketch of first two transitions in terms of Π_N.

$$(\Delta\Pi_N)_I^{II} = \frac{(\Delta Ra_c)_I^{II}}{1+Pr^{-1}} \tag{9}$$

and

$$(\Delta Ra_c)_I^{II} = (Ra_c)_{II} - (Ra_c)_I \quad Pr \to \infty.$$

For liquid metals, $Pr \ll 1$ and equation (9) is reduced to

$$(\Delta\Pi_N)_I^{II} \to (\Delta Ra_c)_I^{II} Pr$$

which is the tangent of equation (9) between domains I and II shown in Fig. 1. As $Pr \to 0$, all transitions collapse on the first transition which now directly leads to turbulence (Domain I in Fig. 1). For gases, $Pr \sim 1$ and equation (9) now covers a narrow band in the middle of Domain II (g-band). For water, $6 < Pr < 30$, equation (9) continues to apply but covers a wider range than that of gases (w-band). For viscous oils, $10^2 < Pr < \infty$, and equation (9) is reduced to

$$(\Delta\Pi_N)_I^{II} \to (\Delta Ra_c)_I^{II}$$

which is independent of Pr because of the negligible inertial effect (Domain III in Fig. 1). The analytical literature, as well, overlooks the significance of Π_N. Beginning with Malkus and Veronis [12] for free boundaries, and continuing with Schluter, Lortz and Busse [13], Gough, Spiegel and Toomre [14] and Busse [15] for rigid boundaries, a first order inertial effect is incorporated into heat transfer by an expansion in powers of Pr^{-1}

$$\frac{Nu-1}{Ra-Ra_c} = (C_1 + C_2 Pr^{-1} + C_3 Pr^{-2} + \ldots)$$

which can be rearranged, in view of

$$1 - Pr^{-1} + Pr^{-2} - Pr^{-3} + \ldots \equiv (1+Pr^{-1})^{-1}$$

as

$$Nu - 1 \sim \frac{Ra - Ra_c}{1+Pr^{-1}}$$

or,

$$Nu - 1 \sim \Delta\Pi_N.$$

For a specified energy generation, the following dimensional equivalence,

$$k\frac{\Delta T}{l^2} \sim u'''$$

rearranged in terms of

$$\Phi = u'''/\rho c_p$$

yields

$$\Delta T \sim \Phi l^2/a$$

and Π_N, now identified with Π_I, depends on

$$Ra_I = \frac{g\beta\Phi l^5}{\nu a^2}. \tag{10}$$

The next section is devoted to the development of microscales for buoyancy driven turbulent flows in terms of Π_I.

3. MICROSCALES

Following the usual practice, decompose the instantaneous velocity and temperature of a buoyancy driven turbulent flow into a temporal mean (denoted by capital letters) and fluctuations

$$\tilde{u}_i = U_i + u_i \text{ and } \tilde{\theta} = \Theta + \theta$$

and let U_i and Θ be statistically steady. Then, the balance of the mean kinetic energy of velocity fluctuations

$$K = \tfrac{1}{2}\overline{u_i u_i}$$

yields (see, for example, Tennekes and Lumley [16])

$$U_j \frac{\partial K}{\partial x_j} = -\frac{\partial \mathscr{D}_j}{\partial x_j} - \mathscr{P}_\beta + \mathscr{P} - \varepsilon \tag{11}$$

where

$$\mathscr{D}_j = \tfrac{1}{2}\overline{pu_j} + \tfrac{1}{2}\overline{u_i u_i u_j} - 2\nu\overline{u_i s_{ij}}$$

is the transport,

$$\mathscr{P}_\beta = -g_j\overline{u_j\theta}/\Theta_0 \tag{12}$$

is the buoyant production, g_j being vector acceleration of gravity and Θ_0 a characteristic temperature for isobaric ambient

$$\mathscr{P} = -\overline{u_i u_j}S_{ij} \tag{13}$$

is the inertial production, and

$$\varepsilon = 2\nu\overline{s_{ij}s_{ij}} \tag{14}$$

is the viscous dissipation of turbulent energy.

Also, the balance of the root mean square of temperature fluctuations

$$K_\theta = \tfrac{1}{2}\overline{\theta^2}$$

gives

$$U_j \frac{\partial}{\partial x_j}(K_\theta) = -\frac{\partial}{\partial x_j}(\mathscr{D}_\theta)_j + \mathscr{P}_\theta - \varepsilon_\theta \tag{15}$$

where

$$(\mathscr{D}_\theta)_j = \tfrac{1}{2}\overline{\theta^2 u_j} - a\frac{\partial}{\partial x_j}(\tfrac{1}{2}\overline{\theta^2})$$

is the thermal transport

$$\mathscr{P}_\theta = -\overline{u_j\theta}\frac{\partial \Theta}{\partial x_j} \tag{16}$$

is the thermal production and

$$\varepsilon_\theta = a\overline{\frac{\partial \theta}{\partial x_j}\frac{\partial \theta}{\partial x_j}} \tag{17}$$

is the thermal dissipation.

For a homogeneous pure shear flow (in which all averaged quantities except U_i and Θ are independent of position and in which S_{ij} and $\partial\Theta/\partial x_j$ are constant), equations (11) and (15) reduces to

$$\mathscr{P}_\beta = \mathscr{P} + (-\varepsilon) \tag{18}$$

and

$$\mathscr{P}_\theta = \varepsilon_\theta. \tag{19}$$

Equation (18) states that the buoyant production is partly converted into inertial production and partly into viscous dissipation.

On dimensional grounds, assuming $S_{ij} \sim u/l$ and $\partial\Theta/\partial x_j \sim \theta/l$, equations (18) and (19) may be written as

$$\mathscr{P}_\beta \sim \frac{u^3}{l} + \nu\frac{u^2}{\lambda^2} \tag{20}$$

and

$$u\frac{\theta^2}{l} \sim a\frac{\theta^2}{\lambda_\theta^2} \tag{21}$$

where u and θ respectively denote the rms values of velocity and temperature fluctuations, l is an integral scale, λ and λ_θ are Taylor scales [17]. Equations (20) and (21) imply isotropic mechanical and thermal dissipations. Note that the isotropic dissipation is usually a good approximation for any turbulent flow (see for example, Tennekes and Lumley [16]).

3.1. *Thermal scales*

To proceed further, invoke the Squire postulate and let

$$\lambda \sim \lambda_\theta \tag{22}$$

in equation (20). This is an often misinterpreted pivotal assumption. It postulates the secondary importance of $\lambda \neq \lambda_\theta$ for heat transfer rather than suggesting equal thickness for scales. The difference in these scales will be illustrated in the subsection on kinetic scales. Now, elimination of velocity between equations (20) and (21) results in a thermal Taylor scale arranged relative to viscous dissipation

$$\lambda_\theta \sim l^{1/3}\left(1+\frac{1}{\sigma}\right)^{1/6}\left(\frac{\nu a^2}{\mathscr{P}_\beta}\right)^{1/6} \tag{23}$$

or, rearranged relative to inertial production,

$$\lambda_\theta \sim l^{1/3}(1+\sigma)^{1/6}\left(\frac{a^3}{\mathscr{P}_\beta}\right)^{1/6} \tag{24}$$

where equation (23) explicitly includes the limit for $\sigma \to \infty$ and is convenient for fluids with $\sigma \geqslant 1$, and equation (24) explicitly includes the limit for $\sigma \to 0$ and is convenient for fluids with $\sigma \leqslant 1$.

For the isotropic flow, replacing both l and λ_θ with one scale, say η_θ,

$$\begin{pmatrix}\lambda_\theta \\ l\end{pmatrix} \to \eta_\theta \tag{25}$$

equations (23) and (24) are respectively reduced to a thermal Kolmogorov scale for buoyancy driven flows†

$$\eta_\theta \sim \left(1+\frac{1}{\sigma}\right)^{1/4}\left(\frac{\nu a^2}{\mathscr{P}_\beta}\right)^{1/4} \tag{26}$$

and

$$\eta_\theta \sim (1+\sigma)^{1/4}\left(\frac{a^3}{\mathscr{P}_\beta}\right)^{1/4}. \tag{27}$$

For $\sigma \gg 1$, equation (26) is reduced to

†The first numeral 1 in the right-hand side of equations (23), (24), (26) and (27) is related to the numeral 1 of equations (3) and (7) and implies order of magnitude.

$$\lim_{\sigma\to\infty} \eta_\theta \to \left(\frac{\nu a^2}{\mathscr{P}_\beta}\right)^{1/4}. \quad (28)$$

Also,

$$\lim_{\sigma\to\infty} \mathscr{P} \to 0 \quad (29)$$

and, in view of equation (18),

$$\mathscr{P}_\beta \sim \varepsilon \quad (30)$$

and equation (28) becomes the scale introduced by Batchelor [18] :

$$\lim_{\sigma\to\infty} \eta_\theta \to \eta_\theta^{\mathrm{B}} \sim \left(\frac{\nu a^2}{\varepsilon}\right)^{1/4}. \quad (31)$$

For $\sigma \ll 1$, equation (26) is reduced to

$$\lim_{\sigma\to 0} \eta_\theta \to \left(\frac{a^3}{\mathscr{P}_\beta}\right)^{1/4}. \quad (32)$$

Also

$$\lim_{\sigma\to 0} \varepsilon \to 0 \quad (33)$$

and, in view of equation (18),

$$\mathscr{P}_\beta \to \mathscr{P}. \quad (34)$$

Then, in a viscous layer order of magnitude thinner than η_θ,

$$\mathscr{P} \to \varepsilon. \quad (35)$$

Now, the inner limit of equation (34) matched to the outer limit of equation (35) leads to equation (30), and equation (32) becomes the scale proposed by Oboukhov [19] and Corrsin [20],

$$\lim_{\sigma\to 0} \eta_\theta \to \eta_\theta^{\mathrm{C}} \sim \left(\frac{a^3}{\varepsilon}\right)^{1/4}. \quad (36)$$

Finally, for $\sigma \sim 1$, because of (an order of magnitude) equipartition of the buoyant production into inertial production and viscous dissipation, equation (18) becomes

$$\mathscr{P}_\beta \sim 2\varepsilon \quad (37)$$

and equations (26) and (27) are reduced to the scale originated by Kolmogorov [21] :

$$\lim_{\sigma\to 1} \eta_\theta = \eta \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}. \quad (38)$$

The relation between the thermal microscales and the integral scale may now be obtained by eliminating the factor $(1+1/\sigma)$ $(\nu a^2/\mathscr{P}_\beta)$ between equations (23) and (26). This readily yields

$$\left(\frac{\eta_\theta}{\lambda_\theta}\right)^2 \sim \frac{\lambda_\theta}{l}. \quad (39)$$

Equations (24) and (27) lead to the same relation, as expected. The foregoing scales are utilized in the next section on the development of a heat transfer correlation for buoyancy driven flows. Before this development, however, the relations between these scales and the dimensionless number Π_{I} need to be shown.

Note that $\mathscr{P}_\beta$ usually depends on velocity, and equation (26) or (27) expressed in terms of velocity cannot be ultimate forms of the Kolmogorov scale for buoyancy driven flows. To eliminate any velocity dependence, reconsider equation (12). On dimensional grounds,

$$\mathscr{P}_\beta \sim gu\theta/\Theta_0. \quad (40)$$

Noting

$$\Theta_0^{-1} \sim \beta$$

β being the coefficient of thermal expansion, rearrange equation (40) as

$$\mathscr{P}_\beta \sim g\beta u\theta \quad (41)$$

or, with the isotropic velocity

$$u \sim a/\eta_\theta \quad (42)$$

obtained from equations (21) and (25), as

$$\mathscr{P}_\beta \sim ga\beta\theta/\eta_\theta. \quad (43)$$

Now, assume θ across η_θ of volume $(\eta_\theta l^2)$ be a result of the rate of internal energy u''' generated per unit of l^3-volume,

$$k\frac{\theta}{\eta_\theta^2}(\eta_\theta l^2) \sim u''' l^3 \quad (44)$$

which gives

$$\theta \sim \left(\frac{\eta_\theta l}{a}\right)\Phi \quad (45)$$

where $\Phi = u'''/\rho c_{\mathrm{p}}$. Elimination of θ between equations (43) and (45) yields

$$\mathscr{P}_\beta \sim g\beta\Phi l. \quad (46)$$

Then, equations (26) and (27) respectively lead to

$$\eta_\theta \sim \left(1+\frac{1}{\sigma}\right)^{1/4}\left(\frac{\nu a^2}{g\beta\Phi l}\right)^{1/4} \quad (47)$$

and

$$\eta_\theta \sim (1+\sigma)^{1/4}\left(\frac{a^3}{g\beta\Phi l}\right)^{1/4} \quad (48)$$

or,

$$\frac{\eta_\theta}{l} \sim \Pi_{\mathrm{I}}^{-1/4} \quad (49)$$

where

$$\Pi_{\mathrm{I}} \sim \frac{Ra_{\mathrm{I}}}{1+Pr^{-1}} = \frac{PrRa_{\mathrm{I}}}{1+Pr}. \quad (50)$$

Two limits of equation (50) are

$$\lim_{Pr\to 0} \Pi_{\mathrm{I}} \to PrRa_{\mathrm{I}} \quad (51)$$

$$\lim_{Pr \to \infty} \Pi_\mathrm{I} \to Ra_\mathrm{I}$$

where

$$Ra_\mathrm{I} = \frac{g\beta}{\nu a}\left(\frac{\Phi l^2}{a}\right) l^3 = \frac{g\beta \Phi l^5}{\nu a^2} \tag{10}$$

is the Rayleigh number based on Φ. Also, from equations (39) and (49),

$$\frac{\lambda_\theta}{l} \sim \Pi_\mathrm{I}^{-1/6}. \tag{52}$$

The following subsection is devoted to the kinetic microscales of buoyancy driven flows.

3.2. *Kinetic scales*

Except for gases, the kinetic scales are markedly different than the foregoing thermal scales. For $\sigma \gg 1$ (viscous oils) the kinetic scale is order of magnitude larger than the thermal scale. That is, the flow extends far beyond the influence of buoyancy ; it is basically isothermal and, in the limit of isotropic flow, is governed by the usual Kolmogorov scale. Also, in this case, the inertial production is negligible, equation (18) is reduced to equation (30), and

$$\eta \sim \left(\frac{\nu^3}{\mathscr{P}_\beta}\right)^{1/4}. \tag{53}$$

For $\sigma \ll 1$ (liquid metals), the kinetic scale is order of magnitude smaller than the thermal scale. The buoyant production within η is negligible, and equation (18) is reduced to equation (35). Also, the viscous dissipation within $\eta_\theta - \eta$ is negligible, and equation (18) is reduced to equation (34). Then, the outer limit of equation (35) matched to the inner limit of equation (34) leads to equation (30), and equation (53) continues to describe the isotropic kinetic scale over a domain between η and η_θ. Now, the ratio between equation (53) and (26) or (27) gives

$$\frac{\eta}{\eta_\theta} \sim \frac{\sigma^{3/4}}{(1+\sigma)^{1/4}} \tag{54}$$

which, for $\sigma \to 0$, is reduced to

$$\lim_{\sigma \to 0}\left(\frac{\eta}{\eta_\theta}\right) \sim \sigma^{3/4} \tag{55}$$

and, for $\sigma \to \infty$ is reduced to

$$\lim_{\sigma \to \infty}\left(\frac{\eta}{\eta_\theta}\right) \sim \sigma^{1/2}. \tag{56}$$

In the next section a heat transfer model based on the foregoing microscales is proposed for buoyancy driven turbulent flows driven by internal energy generation.

4. A HEAT TRANSFER MODEL

Consider a buoyant flow driven by internal energy generated between two horizontal plates. Assume large enough energy generation resulting in fully developed turbulent conditions. This is an ideal problem for a test on the proposed microscales because of the availability of some experimental and analytical literature. In a manner similar to the Prandtl–Taylor two-layer turbulence model for forced convection, let the buoyancy driven turbulent flow be described by a sublayer next to each plate and a core between these layers. Assume each sublayer thickness be characterized by the Kolmogorov scale, and the diffusion (and the intermittent dissipation) in the core by the Taylor scale.

The mean heat flux in the sublayer, in view of the assumed isotropy [recall equation (42)], is

$$\theta \sim k\frac{\theta}{\eta_\theta} \sim \rho c_\mathrm{p} u\theta \tag{57}$$

which shows the same order of magnitude contributions from conduction and convection. The mean heat flux in the core is

$$q_\mathrm{c} \sim k\frac{\theta_\mathrm{c}}{\lambda_\theta} + \rho c_\mathrm{p} u_\mathrm{c}\theta_\mathrm{c} \tag{58}$$

which, in view of equation (21), or

$$\frac{1}{\lambda_\theta} \sim \left(\frac{\lambda_\theta}{l}\right)\frac{u_\mathrm{c}}{a} \tag{59}$$

may be rearranged as

$$q_\mathrm{c} \sim \rho c_\mathrm{p}\left(1 + \frac{\lambda_\theta}{l}\right) u_\mathrm{c}\theta_\mathrm{c} \tag{60}$$

where the subscript c indicates to the core. Then, in view of $\lambda_\theta/l \ll 1$, equation (60) is reduced to

$$q_\mathrm{c} \sim \rho c_\mathrm{p} u_\mathrm{c}\theta_\mathrm{c}. \tag{61}$$

At the interface between the sublayer and core

$$q \sim q_\mathrm{c}. \tag{62}$$

There is conclusive evidence about a temperature reversal in the core of the turbulent Benard problem demonstrated experimentally by Thomas and Townsend [22], Gille [23], and numerically by Herring [24] and Elder [25]. Some of the Kulacki and Emara [7] data on electrolytically heated water indicates also to a similar trend for the present case. Accordingly, in terms of the temperature profile sketched in Fig. 2,

$$\theta - \theta_\mathrm{c} \sim \Delta T \tag{63}$$

where ΔT is the temperature difference between one of the plates and the middle plane. Inserting θ of equation (57) and θ_c of equation (61) into equation (63), noting equation (62),

$$q(1 - a/u_\mathrm{c}\eta_\theta) \sim k\Delta T/\eta_\theta \tag{64}$$

which may be rearranged in terms of the Nusselt number depending on Φ (note $q \sim u''' l \sim \Phi \rho c_\mathrm{p} l$)

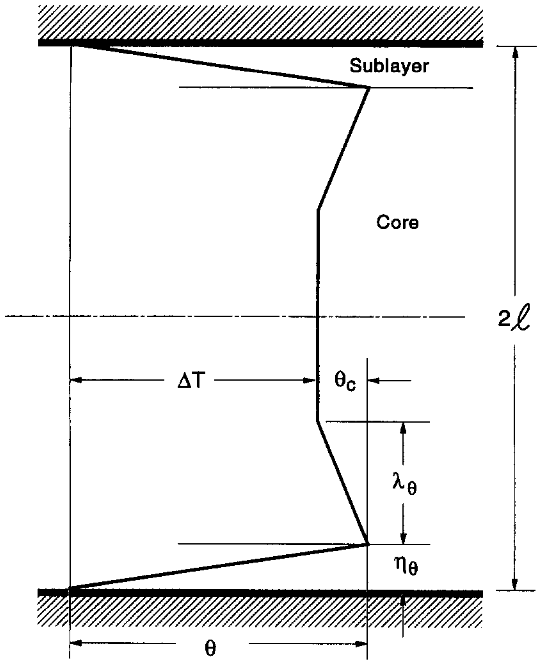

Fig. 2. A sketch of core temperature reversal.

$$Nu = \frac{q}{k(\Delta T/l)} \sim \frac{\Phi l^2}{a\Delta T} \tag{65}$$

as

$$Nu \sim \frac{l/\eta_\theta}{1-(l/\eta_\theta)(u_c l/a)^{-1}} \tag{66}$$

where the numerator shows the contribution of the sublayer and the denominator shows that of the core on heat transfer. To express equation (66) in terms of the length scales alone, reconsider equation (21) for velocity of the core,

$$u_c \sim a\frac{l}{\lambda_\theta^2} \tag{67}$$

which may be rearranged as

$$\frac{u_c l}{a} \sim \left(\frac{l}{\lambda_\theta}\right)^2. \tag{68}$$

In terms of this relation, equation (66) becomes

$$Nu \sim \frac{l/\eta_\theta}{1-(l/\eta_\theta)(l/\lambda_\theta)^{-2}} \tag{69}$$

which, in view of equations (49) and (52), yields a model for any Prandtl number

$$Nu \sim \frac{\Pi_I^{1/4}}{1-\Pi_I^{-1/12}}. \tag{70}$$

The two limits of this result,

$$\lim_{Pr\to 0} Nu \sim \frac{(PrRa_I)^{1/4}}{1-(PrRa_I)^{-1/12}} \tag{71}$$

and

$$\lim_{Pr\to\infty} Nu \sim \frac{Ra_I^{1/4}}{1-Ra_I^{-1/12}}, \tag{72}$$

are identical to the models already proposed by Cheung [5]. Thus, the present study generalizes, via microscales appropriate for buoyancy driven flows, two Cheung correlations into equation (70) which is valid for fluids of any Prandtl number. Now, equation (70) may be written as an equality in terms of three constants

$$Nu = \frac{C_1\Pi_I^{1/4}}{1-C_2\Pi_I^{-1/12}} \quad \Pi_I = \left(\frac{Pr}{C_0+Pr}\right)Ra_I \tag{73}$$

and provides a heat transfer correlation for turbulent natural convection driven by internal energy generation between two parallel plates. Although the values of C_0, C_1 and C_2 must be determined from experimental data, they are expected to be numerical constants.

The experimental literature on the buoyant turbulent flow driven by volumetric internal energy generation is confined to the studies of Tritton and Zarraga [26], Fiedler and Wille [27], Kulacki and Emara [7] and Kulacki and Nagle [8]. These studies employ electrolytically heated water for which Pr remains within the narrow range of 6–7. If one assumes $C_0 \ll 1$ indicating to a small inertial effect (see Arpaci [1]),

$$\Pi_I \to Ra_I\,Pr > 1$$

and Nu given by equation (73) is reduced to

$$Nu = \frac{C_1 Ra_I^{1/4}}{1-C_2 Ra_I^{-1/12}}. \tag{74}$$

Cheung [5] employs the data of Kulacki and Emara and proposes

$$Nu = \frac{0.206 Ra_I^{1/4}}{1-0.847 Ra_I^{-1/12}}. \tag{75}$$

Figure 3 taken from Cheung shows the correlation of the experimental data by equation (75). A correlation for any Prandtl number involving the numerical values of C_0 and C_1 in equation (73) needs data for another Prandtl range (preferably for liquid metals) which is not presently available. However, for buoyant turbulent flows between two horizontal plates kept at different temperatures, there is extensive data for a variety of fluids (including liquid metals, gases, water and viscous oils). A recently proposed model by Arpaci [1], Arpaci and Dec [28]

$$Nu = \frac{0.0471\Pi_N^{1/3}}{1-1.734\Pi_N^{-1/9}} \quad \Pi_N = \frac{Ra}{1+0.0414Pr^{-1}}$$

correlates these data in terms of Π_N over the range of 10^6–10^{11}.

5. CONCLUDING REMARKS

For buoyant turbulent flows driven by internal energy generation, a fundamental dimensionless num-

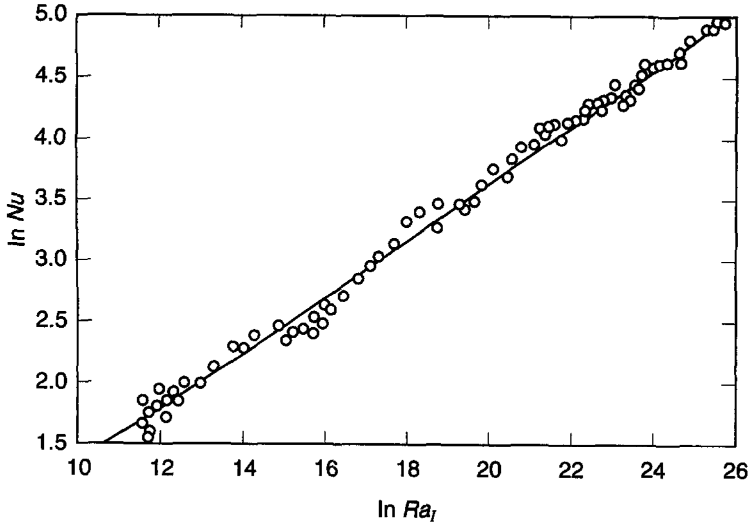

Fig. 3. ln Nu vs ln Ra_I. ——, Cheung model given by equation (75); ○, data of Kulacki and Emara (1977).

ber involving a combination of Prandtl and Rayleigh numbers is proposed. Thermal and kinetic microscales appropriate for these flows are then developed in terms of this number. A heat transfer model for buoyant turbulent flow between two horizontal plates is constructed in terms of these scales. The model incorporates both sublayer and core effects, includes inertial effects, and applies to all fluids (for any Prandtl number). Consequently, it generalizes the previously proposed two Cheung models into one. In the appendix, the Cheung study is interpreted under the light of the proposed scales.

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APPENDIX

Consider equation (15) of Cheung [5],

$$\delta_t \sim (g\beta)^{-1/4} v^{3/4} (\Phi l)^{-1/4} Pr^{-1/2}$$

and rearrange it as

$$\delta_t \sim \left(\frac{v^3}{g\beta\Phi l}\right)^{1/4} Pr^{-1/2} \tag{A1}$$

or, as

$$\delta_t \sim \left(\frac{va^2}{g\beta\Phi l}\right)^{1/4} \tag{A2}$$

which turns out to be the large Prandtl limit of equation (47) and is a Batchelor scale. Next, combine equation (25) of Cheung,

$$\delta_t/\delta_v \sim Pr^{-1/2}$$

with equation (A1), to get

$$\delta_v \sim \left(\frac{v^3}{g\beta\Phi l}\right)^{1/4} \tag{A3}$$

which is a Kolmogorov scale. Finally, rearrange equation (49) of Cheung,

$$\delta_t/l \sim Pr^{-1/4} Ra_l^{-1/4}$$

as

$$\delta_t \sim \left(\frac{a^3}{g\beta\Phi l}\right)^{1/4} \tag{A4}$$

which is the small Prandtl limit of equation (48) and is an Oboukhov–Corrsin scale. Now, the entire Cheung study can be interpreted in terms of equations (A2), (A3) and (A4).